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# Vacuum Polarisation Effects in the Lorentz Violating Electrodynamics

**G. Bonneau · L.C. Costa · J.L. Tomazelli**

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**Abstract** In this work we reconsider the one loop calculation for the vacuum polarisation tensor in the Lorentz violating quantum electrodynamics. The electron propagator is "dressed" by a Lorentz breaking extra term in the fermion Lagrangian density. We check gauge invariance and use the Schwinger–Dyson equation to discuss the full photon propagator. After a discussion on a possible photon mass shift, we show how a finite quantum correction can be chosen in a unique way in order to ensure—in the spirit of spontaneously broken theories—the standard normalisation conditions for the vacuum polarisation tensor. Then we comment on possible observable physical consequences on the Lamb-shift.

**Keywords** Radiative corrections · CPT and Lorentz violation · Chern–Simons

## 1 Introduction

The PCT theorem states that if a field theory satisfies the following axioms: (a) *locality*, (b) *Lorentz invariance* and (c) *analyticity of the Lorentz group representation in the boost*

G. Bonneau
Laboratoire de Physique Théorique et Hautes Energies, Universités de Paris VI et Paris VII,
Unité associe au CNRS UMR 7589, 2 Place Jussieu, 75251 Paris Cedex 05, France
e-mail: bonneau@lpthe.jussieu.fr

L.C. Costa (✉)
Departamento de Física, Universidade Federal de Goiás, Campus de Catalão,
Av. Lamartine P. Avelar, 1120, 75704-020 Catalão, GO, Brazil
e-mail: luciocamposcosta@click21.com.br

L.C. Costa
International Center for Condensed Matter Physics, Universidade de Brasília, Campus Darcy Ribeiro,
70904-910 Brasília, DF, Brazil

J.L. Tomazelli
Departamento de Física, Universidade Federal de Santa Catarina, Campus Universitário,
88010-970 Florianópolis, SC, Brazil
e-mail: tomazelli@fsc.ufsc.br

*parameters*, the PCT transformation is a symmetry of the theory itself [1]. In this context, the invariance under the Lorentz group represents one of the fundamental axioms in the construction of a relativistic quantum field theory, which includes the minimal $SU(3) \times SU(2) \times U(1)$ Standard Model.

The possibility that the physical world slightly violates the Lorentz and PCT symmetries has been the object of intense research activities in different areas of physics, ranging from Quantum Optics to Neutrino Physics [2]. However, up to now, there has been no conclusive experimental evidence for the violation of axiom (b).

From the theoretical standpoint, there arose a controversy on a possible Chern–Simons-like term generated through radiative corrections in an extended version of Quantum Electrodynamics (QED) [3–10]. In fact, some authors claim that the addition to the QED Lagrangian density of a power counting renormalizable and Lorentz violating term such as

$$\mathcal{L}_{LB} = -b^{\alpha}\bar{\psi}\gamma_{\alpha}\gamma_5\psi \tag{1}$$

induces a Chern–Simons-like term, namely,

$$\mathcal{L}_{CS} = \frac{1}{2}c_{\mu}\epsilon^{\mu\alpha\beta\gamma}F_{\alpha\beta}A_{\gamma}, \tag{2}$$

where $b^{\alpha}$ is a set of four constants, which selects a preferencial direction in space–time, thus violating Lorentz invariance; $\psi$ represents the electron field; $\gamma_5$ a hermitian matrix with the properties $\{\gamma_5, \gamma_{\alpha}\} = 0$ and $\mathrm{tr}\gamma_5\gamma^{\alpha}\gamma^{\beta}\gamma^{\gamma}\gamma^{\delta} = 4i\epsilon^{\alpha\beta\gamma\delta}$. *Note that such a term* (2) *does not destroy the gauge invariance of the action. Note also that, as explained in* [9], *even if that term destroy the gauge invariance of the Lagrangian, it acts as a minor modification of the gauge fixing term as $\partial_{\mu}A^{\mu}$ remains a free field.*

In this context some issues arise:

- As soon as Lorentz symmetry is violated, one wonders about the validity of any calculation: should one consider—as done for example in [5]—that the loop momenta phase space is no longer Lorentz covariant? Should the regularisation preserve Lorentz invariance? Here, we will follow the point of view initiated by Colladay and Kostelecky in which the sole breaking of Lorentz invariance comes from the added Lagrangian (1). This corresponds to the spirit of a theory where Lorentz invariance is spontaneously broken (SSBL) [2, 11–14].
- The Lagrangian (1) modifies the fermion propagator and two approaches are then possible. The first one is an expansion in $b^{\alpha}$ of the propagator which amounts to consider (1) as a new interaction: we call this approch a perturbative one. Alternatively, one can consider calculations with the complete $b^{\alpha}$ dependent fermion propagator: we call this approach a non-perturbative one (see [15] for a discussion on these two approaches).

Thus, if the Lorentz breaking is of the SSBL type and if the electron propagator is expanded in powers of $b^{\alpha}$, and moreover if the theory is correctly defined through Ward identities and normalisation conditions, no Chern–Simons term seems to appear [9].[1,2] A similar

[1] As shown in [9], this is due to the fact that such a term (2) does not modify the Ward identity that constrains the vacuum polarisation tensor. Another way of saying this is that such a term, being a kind of minor modification of the gauge fixing term, is then not renormalised: as a consequence, if absent at the classical level, it remains absent at the loop level.

[2] As also implied in [16], the finiteness of an induced Chern–Simons term and its non-renormalisation is related to the well known non-renormalisation theorem of the axial anomaly.

result was first obtained in the analysis due to Coleman and Glashow [3] and this result was afterwards confirmed through a proper-time approach by Sytenko and Rulik [10].

In this work we intend to examine, at the one-loop level, the consequences of a spontaneous breaking of Lorentz invariance on standard QED (a gauge invariant theory with a massless photon[3]). So, we compute the vacuum polarisation amplitude $\Pi^{\mu\nu}$ in QED (regulated by means of the gauge invariant Pauli–Villars–Rayski (P–V–R) scheme [17, 18]) when the tiny Lorentz breaking term (1) is added. Note that—thanks to the non-renormalisation theorem of [9]—and in accordance with experimental results, we do not introduce a term (2) at the classical level: of course, this corresponds to the specific normalisation condition on the antisymmetric part of the vacuum polarisation tensor:

$$\epsilon^{\mu\nu\rho\sigma}\frac{\partial}{\partial p^{\rho}}\langle A_{\mu}(p)\,A_{\nu}(-p)\rangle^{proper}\bigg|_{p=0}=0. \tag{3}$$

In Sect. 2 we consider (1) as an interaction term and then do a double expansion in $b_\alpha$ and $\hbar$. We give the complete one loop calculation to second order in the breaking parameter $b_\alpha$. In order to compare with previous calculations [19–21] we give the results for every one-loop contributions. As expected from general results [9], we check that gauge invariance is maintained. However, in a first approach, a new pole seem to appear in the photon propagator. Then we argue that the usual normalisation condition for the vacuum polarisation tensor should be preferred and in Sect. 3 we show that a uniquely defined quantum correction does the job.

The full photon propagator is obtained by summing the perturbative series through the Schwinger–Dyson equation [22, 23], permitting a detailed discussion on the masslessness of the photon in that theory. Consequences for the Lamb-shift are also addressed. Finally some concluding remarks are offered in Sect. 4.

## 2 The One-Loop Vacuum Polarisation Tensor in Extended QED

In the extended version of QED, defined by the Lagrangian density

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2\alpha}(\partial_{\mu}A^{\mu})^2+\bar{\psi}(i\not\partial-e\not A-\not b\gamma_5-m)\psi, \tag{4}$$

the fermion propagator is

$$S(l)=\frac{i}{\not l-m-\not b\gamma_5}=\sum_{n=0}^{\infty}\frac{i}{\not l-m}\left\{-i\,\not b\gamma_5\frac{i}{\not l-m}\right\}^n=\sum_{n=0}^{\infty}S_n(l).$$

The one-loop vacuum polarisation tensor is

$$\Pi^{\mu\nu}(p,m,b)=-(-i\,e)^2\int\frac{d^4l}{(2\pi)^4}\,\mathrm{Tr}[\gamma^{\mu}S(l)\gamma^{\nu}S(l+p)]. \tag{5}$$

[3]This masslessness is expressed through the usual normalisation condition on the unbroken part of the vacuum polarisation tensor at vanishing squared momentum:

$$\Pi^{\mu\nu}(p,-p)=[p^{\mu}p^{\nu}-g^{\mu\nu}p^2]\Pi(p^2),\quad \Pi(0)=0.$$

$\Pi^{\mu\nu}(p,m,b)$ then admits the decomposition

$$\Pi^{\mu\nu}(p,m,b) = \Pi_0^{\mu\nu}(p,m) + \Pi_b^{\mu\nu}(p,m,b) + \Pi_{bb}^{\mu\nu}(p,m,b). \tag{6}$$

In the last expression,

$$\Pi_0^{\mu\nu}(p,m) = e^2 \int \frac{d^4 l}{(2\pi)^4} \operatorname{Tr}[\gamma^\mu S_0(l)\gamma^\nu S_0(l+p)]$$

is the usual QED vacuum polarisation tensor,

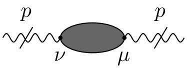
$$\Pi_b^{\mu\nu}(p,m,b) = e^2 \int \frac{d^4 l}{(2\pi)^4} \operatorname{Tr}[\gamma^\mu S_0(l)\gamma^\nu S_1(l+p) + \gamma^\mu S_1(l)\gamma^\nu S_0(l+p)] \tag{7}$$

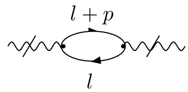
is the linear $b_\alpha$ contribution, and

$$\begin{aligned}\Pi_{bb}^{\mu\nu}(p,m,b) = {} & e^2 \int \frac{d^4 l}{(2\pi)^4} \operatorname{Tr}[\gamma^\mu S_1(l)\gamma^\nu S_1(l+p)] \\ & + e^2 \int \frac{d^4 l}{(2\pi)^4} \{\operatorname{Tr}[\gamma^\mu S_2(l)\gamma^\nu S_0(l+p)] + \operatorname{Tr}[\gamma^\mu S_0(l)\gamma^\nu S_2(l+p)]\}\end{aligned} \tag{8}$$

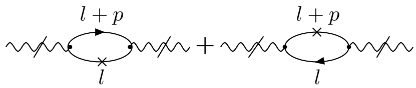
is the $\mathcal{O}(b^2)$ contribution to the polarisation tensor, whose thorough calculation will be one of the focuses of the present work.

Diagrammaticaly, the expansion (6) may be represented by the series

$\Pi^{\mu\nu}(p,m,b) \equiv$ 


$=$ 


*"graph with no b-insertion"*

$+$ 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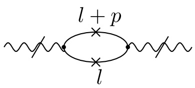
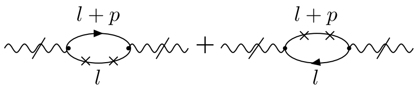
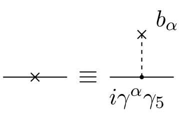
$+$ 


*"graph with two b-insertions: (1–1)"*

$+$ 


*"graphs with two b-insertions: respectively (2–0) and (0–2)"*

where



represents the linear $b_\alpha$ insertion to the (internal) fermion lines.

By power counting, $\Pi^{\mu\nu}_{bb}(p,m,b)$, $\Pi^{\mu\nu}_{b}(p,m,b)$ and $\Pi^{\mu\nu}_{0}(p,m)$ are respectively logarithmicaly, linearly and quadratically divergent in the ultra-violet region. Of course, a regularisation procedure is needed: in order to preserve as much as possible the symmetries of the classical theory (4) (*i.e.*, gauge invariance), the Pauli–Villars–Rayski (P–V–R) regularisation prescription will be employed [17, 18]. In this scheme, auxiliary fermion masses satisfying specific conditions are introduced. The original theory is recovered at the end of the calculations, by taking arbitrary large values for the auxiliary masses.

Let us recall also that thanks to gauge invariance, the degree of divergence of the tensor $\Pi^{\mu\nu}_0(p)$, identical to the one of the pure QED sector, is reduced from 2 to 0 (logarithmic divergence).

To be consistent with the P–V–R scheme, the tensor $\Pi^{\mu\nu}(p,m,b)$ must be regulated as a whole object, which implies the replacement of expression (6) by the sum

$$\Pi^{\mu\nu}(p,b)=\sum_{i=0}^{N}c_i(\Pi^{\mu\nu}_0(p,m_i)+\Pi^{\mu\nu}_b(p,m_i,b)+\Pi^{\mu\nu}_{bb}(p,m_i,b)), \tag{9}$$

where each term in the r.h.s. of (9) retains the original functional form, except for the fact that $m\to m_i$ (note also that we consider the same Lorentz breaking parameter $b^\alpha$ for all fermions). By analysing the structure of (9), it can be shown that, for auxiliary masses satisfying the conditions:

$$\begin{aligned}(a)\quad &\sum_{i=0}^{N}c_i=0,\\ (b)\quad &\sum_{i=0}^{N}c_i m_i^2=0,\end{aligned} \tag{10}$$

all the divergences in $\Pi^{\mu\nu}(p,m,b)$ disappear. In the above expressions, $c_0=1$ and $m_0=m$ is the electron mass.

The results may be written as

$$\Pi^{\mu\nu}_0(p,m_i)=[g^{\mu\nu}p^2-p^\mu p^\nu]\Pi_0(p^2,m_i), \tag{11}$$

$$\Pi^{\mu\nu}_b(p,m_i,b)=\epsilon^{\mu\nu\alpha\beta}p_\alpha b_\beta\Pi_b(p^2,m_i), \tag{12}$$

$$\Pi^{\mu\nu}_{bb}(p,m_i,b)=\Pi^{\mu\nu(1-1)}_{bb}(p,m_i,b)+\Pi^{\mu\nu(0-2)}_{bb}(p,m_i,b), \tag{13}$$

$$\begin{aligned}\Pi^{\mu\nu(j-k)}_{bb}(p,m_i,b)=&\ \mathbb{A}^{(j-k)}((p.b)^2,b^2,p^2,m_i)g^{\mu\nu}+\mathbb{B}^{(j-k)}(p^2,m_i)b^\mu b^\nu\\ &+\mathbb{C}^{(j-k)}(p^2,m_i)(p.b)(b^\mu p^\nu+b^\nu p^\mu)\\ &+\mathbb{D}^{(j-k)}((p.b)^2,b^2,p^2,m_i)p^\mu p^\nu,\end{aligned} \tag{14}$$

where the superscript (1–1) (resp. (0–2)) refers to graph (1–1) (resp. graphs (0–2) and (2–0)) and

$$\Pi_0(p^2,m_i)=i\frac{e^2}{12\pi^2}\left\{\log\frac{m_i^2}{m^2}-p^2\int_0^1 dz\,\frac{[1-2Z-8Z^2]}{2\Delta_i}\right\}, \tag{15}$$

$$\Pi_b(p^2,m_i)=i\frac{e^2}{2\pi^2}\left\{p^2\int_0^1 dz\,\frac{Z}{\Delta_i}\right\}, \tag{16}$$

$$\mathbb{A}^{(1-1)}((p.b)^2, b^2, p^2, m_i) = -i\frac{e^2}{\pi^2}\left\{\frac{b^2}{6}\log\frac{m_i^2}{m^2} + \int_0^1 dz\left[\frac{-1-4Z+8Z^2}{12\Delta_i}\, b^2 p^2 + \frac{Z^2}{\Delta_i}(b.p)^2 + \frac{Z^2}{2[\Delta_i]^2}p^2[(b\cdot p)^2 - b^2p^2]\right]\right\}, \tag{17}$$

$$\mathbb{B}^{(1-1)}(p^2, m_i) = -i\frac{e^2}{\pi^2}\left\{-\frac{1}{6}\log\frac{m_i^2}{m^2} + p^2\int_0^1 dz\left[\frac{1+10Z-8Z^2}{12\Delta_i} + \frac{Z^2}{[\Delta_i]^2}p^2\right]\right\}, \tag{18}$$

$$\mathbb{C}^{(1-1)}(p^2, m_i) = -i\frac{e^2}{\pi^2}\left\{\int_0^1 dz\left[-\frac{Z(1-2Z)}{\Delta_i} - \frac{Z^2}{[\Delta_i]^2}p^2\right]\right\}, \tag{19}$$

$$\mathbb{D}^{(1-1)}((p.b)^2, b^2, p^2, m_i) = -i\frac{e^2}{\pi^2}\left\{\int_0^1 dz\left[\frac{3Z^2}{\Delta_i}b^2 + 2\frac{Z^3}{[\Delta_i]^2}((p.b)^2 + b^2p^2)\right]\right\}. \tag{20}$$

$$\mathbb{A}^{(0-2)}((p.b)^2, b^2, p^2, m_i) = -i\frac{e^2}{\pi^2}\left\{-\frac{b^2}{6}\log\frac{m_i^2}{m^2} + \int_0^1 dz\left[\frac{1-8Z-8Z^2}{12\Delta_i}b^2p^2 + \frac{4Z^2}{\Delta_i}(b.p)^2 + \frac{2Z^3}{[\Delta_i]^2}p^2(b\cdot p)^2 - \frac{Z^2}{2[\Delta_i]^2}(p^2)^2b^2\right]\right\}, \tag{21}$$

$$\mathbb{B}^{(0-2)}(p^2, m_i) = -i\frac{e^2}{\pi^2}\left\{\frac{1}{6}\log\frac{m_i^2}{m^2} - \int_0^1 dz\left[\frac{(1-4Z)(1+2Z)}{12\Delta_i}p^2\right]\right\}, \tag{22}$$

$$\mathbb{C}^{(0-2)}(p^2, m_i) = -i\frac{e^2}{\pi^2}\left\{\int_0^1 dz\left[-\frac{2Z^2}{\Delta_i}\right]\right\}, \tag{23}$$

$$\mathbb{D}^{(0-2)}((p.b)^2, b^2, p^2, m_i) = -i\frac{e^2}{\pi^2}\left\{\int_0^1 dz\left[\frac{Z(1-Z)}{2\Delta_i}b^2 + \frac{Z^2(3-4Z)}{4[\Delta_i]^2}b^2p^2 - 2\frac{Z^3}{[\Delta_i]^2}(p.b)^2\right]\right\}. \tag{24}$$

In those expressions, we have set

$$Z = z(1-z),$$
$$\Delta_i = m_i^2 - z(1-z)p^2.$$

Moreover, to simplify our results, we used some integration by parts on $z$ and the symmetry of the integral when $z$ is changed into $(1-z)$: this allows us to rewrite the integrands as functions of $Z$.

The full tensor $\Pi_{bb}^{\mu\nu}(p, m, b)$ simplifies to:

$$\Pi_{bb}^{\mu\nu}(p, m, b) = -i\frac{e^2}{\pi^2}Z^{\mu\nu}\int_0^1 dz\left[\frac{Z}{\Delta_i} + \frac{Z^2}{[\Delta_i]^2}p^2\right], \tag{25}$$

$$Z^{\mu\nu} = -X^{\mu\nu} + Y^{\mu\nu}, \tag{26}$$

where we have introduced the two transverse tensors of dimension 4, quadratic in $b^\alpha$:

$$\begin{aligned}&\text{(a)}\quad X^{\mu\nu} = b^2(g^{\mu\nu}p^2 - p^\mu p^\nu),\\ &\text{(b)}\quad Y^{\mu\nu} = g^{\mu\nu}(p.b)^2 + p^2b^\mu b^\nu - (p.b)(p^\mu b^\nu + p^\nu b^\mu).\end{aligned} \tag{27}$$

Notice that $Z^{\mu\nu}$ is the unique transverse tensor of dimension 4, quadratic in $b^\alpha$ and *also transverse with respect to* $b^\mu$. Further, if one had chosen another perturbative algorithm, replacing the fermion propagator by its first order in $b^\alpha$ dressed one (limiting $\Pi_{bb}^{\mu\nu}$ to $\Pi_{bb}^{\mu\nu(1-1)}$), gauge invariance would have been definitely lost [24]. In the same way, once we take into account all contributions to order $b^2$, according to "box" diagrams (1–1), (2–0) and (0–2), we obtained the transversality in the Lorentz-breaking parameter.

At this point, a few comments are in order:

- The scale in the logarithms has been (arbitrarily) chosen to be the electron mass, thanks to the condition (10a) $\sum_{i=0}^{i=N} c_i = 0$.
- With that double expansion in $b^\alpha$ and $\hbar$, gauge invariance holds at the regularized level and then we checked that *no ultra-violet divergence remains* in $\Pi_b^{\mu\nu}$ and in the full $\Pi_{bb}^{\mu\nu}$. Then the solely required "infinite" renormalisation of the vacuum polarisation tensor is the standard QED one

$$-\frac{e^2}{12\pi^2}\left[\sum_{i=0}^{i=N} c_i \log\frac{m_i^2}{m^2}\right]\left\{\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right\}. \tag{28}$$

  Thanks to that gauge invariance, the degree of divergence of the vacuum polarisation tensor, to all orders in $b^\alpha$, is lowered from 2 to 0: then we learn from general principles in QFT that one and only one subtraction is necessary to define the finite renormalized vacuum polarisation tensor. In other words, the full one-loop counterterm contains the infinite part (28) plus a finite part that depends on the chosen renormalisation conditions.
- Since $\Pi_b^{\mu\nu}$ vanishes at $p^2 = 0$, no Chern–Simons-like term appears [9]. Moreover, should one add a *finite* counterterm of type (2), the normalisation condition (3) would no longer hold.
- However, $\Pi_{bb}^{\mu\nu}(p, m_i, b)$ does not vanish at $p^2 = 0$. Notice that—thanks to gauge invariance—its degree of divergence is lowered from 2 to $-2$: then it may be obtained through a finite Feynmann integral and, as a consequence, *its value is fully unambiguous (scheme independant) and no normalisation condition is required* (the same would of course be true for higher orders in $b^\alpha$).[4] Denoting as $\Pi_{min}^{(R)\mu\nu}$ the renormalized vacuum polarisation tensor (again, once the QED counterterm has been added, the limit $m_i \to \infty$ may be taken, except for $i = 0$ where $m_0 = m$ and $c_0 = 1$), one gets:

$$\Pi_{min}^{(R)\mu\nu}|_{p^2=0} = -i\frac{e^2}{6\pi^2 m^2}\left[g^{\mu\nu}(p.b)^2 + b^2 p^\mu p^\nu - (p.b)(p^\mu b^\nu + p^\nu b^\mu)\right].$$

  Would one add no finite counter-term (in a kind of minimal subtraction scheme), the nonvanishing of $\Pi_{min}^{(R)\mu\nu}$ at $p^2 = 0$ would indicate that the pole structure of the photon propagator is modified, leading to some mass shift for the photon. To be more precise, we use Schwinger–Dyson summation (see next section for the details on the inversion of the vacuum polarisation tensor) to obtain an expression for the radiatively corrected photon propagator. Expanding for small $p^\alpha$ gives rise to an additional pole

$$\mu^2 \equiv -\frac{e^2}{\pi^2}\frac{(p.b)^2}{3m^2}$$

  which, in principle, would correspond to a massive tachyonic mode; however, as discussed in [25], such a $(p.b)^2$ dependent pole may be circumvented by using the Cauchy

[4] We thank the referee who urges us to justify our unequal treatment of terms of order $b$ and $b^2$.

principal value prescription, or other causal prescriptions such as Mandelstam's and Leibbrandt's ones [26], according to the physical nature of the Lorentz-breaking parameter. Remarkably, one can borrow such procedure from QCD in non-covariant gauges, when tackling the spurious gauge-dependent poles in the gluon propagator [27, 28].

Then we analyse another choice, more in the spirit of spontaneously broken theories: as will be detailed in the next section, we use the *same normalisation condition*

$$\Pi^{(R)\mu\nu}|_{p^2=0} = 0$$

*that define the physical parameters of the unbroken theory* and add the corresponding finite quantum correction to the Lagrangian density.

- Our calculation for the $\Pi^{\mu\nu}_{bb}$ allows us to correct a sign error in a previous calculation (formula (20) of [19]) and, as gauge invariance is maintained, differs from the one obtained in [21].[5]

## 3 The Renormalized Vacuum Polarisation Tensor

According to the previous discussion, we enforce the normalisation condition

$$\Pi^{(R)\mu\nu}|_{p^2=0} = 0, \tag{29}$$

through addition of an extra finite $\mathcal{O}(\hbar)$ term in the Lagrangian. Among those possible ones which compensate for the previous quantity up to $p^2 = 0$ vanishing contributions,

$$\begin{aligned} &i\,\frac{e^2}{6\pi^2 m^2}\left[g^{\mu\nu}(p.b)^2 + b^2 p^\mu p^\nu - (p.b)(p^\mu b^\nu + p^\nu b^\mu)\right] + \mathcal{O}(p^2) \\ &= i\,\frac{e^2}{6\pi^2 m^2}\left[Y^{\mu\nu} - X^{\mu\nu}\right] + \mathcal{O}'(p^2), \end{aligned}$$

we choose the *unique* gauge invariant term:[6]

$$\frac{\hbar e^2}{6\pi^2 m^2}\left[g^{\mu\nu}(b^\alpha F_{\alpha\mu})(b^\beta F_{\beta\nu}) - \frac{b^2}{4} F_{\mu\nu}F^{\mu\nu}\right]. \tag{30}$$

In order to obtain the full photon propagator, the Schwinger–Dyson equation must be solved [22, 23]. In a given order of the perturbative series, it implies the summation of an infinite set of proper Feynman diagrams. In this way, as

$$(G_0{}^{-1})^{\mu\nu} = i\left[p^2 g^{\mu\nu} - \left(1 - \frac{1}{\alpha}\right)p^\mu p^\nu\right]$$

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[5] See a discussion on that disrepancy in [15].

[6] Notice that such Lorentz breaking term was not added to the classical Lagrangian: its presence should modify the classical photon propagator and would require another physical parameter. As previously argued, the unambiguous value of $\Pi^{\mu\nu}_{bb}$ allows for this absence of a specific normalisation condition that would define this parameter. Of course, still to one loop order but in higher orders in the breaking $b^\alpha$, our normalisation condition (29) will also require other finite terms to which the same comment applies. All these $\hbar$ terms being power counting renormalisable and chosen to be gauge invariant, one could go to higher loop order where they would all be renormalised (infinitely)—in the same manner as the charge $e$, the electron mass $m$ and the breaking $b^\alpha$, are (infinitely) renormalised at the one loop level. But this higher loop analysis is out of the scope of our present analysis.

is the inverse of the QED free photon propagator, the Schwinger–Dyson equation

$$(G^{-1})^{\mu\nu} = (G_0^{-1})^{\mu\nu} - \bar{\Pi}^{(R)\mu\nu}(p, m, b)$$

leads us to

$$(G^{-1})^{\mu\nu} = i\left[A' g^{\mu\nu} + B' b^\mu b^\nu + C'(b^\mu p^\nu + b^\nu p^\mu) + D' p^\mu p^\nu + E' \epsilon^{\mu\nu\alpha\beta} p_\alpha b_\beta\right], \quad (31)$$

where the values of the functions $A'$, $B'$, $C'$, $D'$ and $E'$ may be obtained from the results in the previous section and are given below (34).

From (31) we obtain the inverse tensor:

$$G_{\mu\nu} = -i\left[A'' g_{\mu\nu} + B'' b_\mu b_\nu + C''(b_\mu p_\nu + b_\nu p_\mu) + D'' p_\mu p_\nu + E'' \epsilon_{\mu\nu\alpha\beta} p^\alpha b^\beta\right], \quad (32)$$

where[7]

$$\begin{aligned}
A'' &= \frac{A'}{D_1}, \\
B'' &= \frac{p^2 E'^2}{A'[D_1]} - \frac{A'B' + p^2(B'D' - C'^2)}{A'[D_2]}, \\
C'' &= -\frac{(p.b)E'^2}{A'[D_1]} - \frac{A'C' - (p.b)(B'D' - C'^2)}{A'[D_2]}, \\
D'' &= \frac{b^2 E'^2}{A'[D_1]} - \frac{A'D' + b^2(B'D' - C'^2)}{A'[D_2]}, \\
E'' &= -\frac{E'}{D_1}.
\end{aligned} \quad (33)$$

Here

$$\begin{aligned}
A' &= p^2\left\{1 + \frac{e^2}{2\pi^2}\left[p^2\beta(p^2, m^2) + [(p.b)^2 - p^2b^2]\chi(p^2, m^2)\right]\right\}, \\
B' &= \frac{e^2}{2\pi^2}(p^2)^2\chi(p^2, m^2), \\
C' &= -\frac{e^2}{2\pi^2}(p^2)(p.b)\chi(p^2, m^2), \\
D' &= \frac{1-\alpha}{\alpha} + \frac{e^2}{2\pi^2}[-p^2\beta(p^2, m^2) + p^2b^2\chi(p^2, m^2)], \\
E' &= -\frac{e^2}{2\pi^2}p^2\gamma(p^2, m^2),
\end{aligned} \quad (34)$$

[7] We have defined:

$$D_1 = A'^2 + [p^2b^2 - (p.b)^2]E'^2,$$
$$D_2 = A'[A' + b^2B' + 2(p.b)C' + p^2D'] + [p^2b^2 - (p.b)^2](B'D' - C'^2).$$

Note that if $(G^{-1})^{\mu\nu}$ were purely transverse ($A' + (p.b)C' + p^2D' = (p.b)B' + p^2C' = 0$), the inversion would not be possible as $D_2 \equiv 0$.

where:

$$\begin{aligned}\beta(p^2,m^2) &= \int_0^1 dz \frac{[1-2Z-8Z^2]}{12\Delta_0},\\ \chi(p^2,m^2) &= \frac{2}{p^2}\int_0^1 dz\left\{\left[\frac{Z}{\Delta_0}+\frac{Z^2}{[\Delta_0]^2}p^2\right]-\left[\frac{Z}{m^2}\right]\right\}\\ &= 2\int_0^1 dz Z^2\left[\frac{1}{m^2\Delta_0}+\frac{1}{[\Delta_0]^2}\right],\\ \gamma(p^2,m^2) &= \int_0^1 dz\,\frac{Z}{\Delta_0}.\end{aligned} \tag{35}$$

The functions $\beta$, $\chi$ and $\gamma$ are finite at $p^2=0$:

$$\beta(0,m^2)=1/(30m^2), \qquad \chi(0,m^2)=2/(15m^4), \qquad \gamma(0,m^2)=1/(6m^2).$$

We obtain to first order in $e^2$:

$$\begin{aligned}D_1 &\simeq A'^2 \simeq (p^2)^2\left\{1+\frac{e^2}{\pi^2}\left[p^2\beta(p^2,m^2)+[(p.b)^2-p^2b^2]\chi(p^2,m^2)\right]\right\},\\ D_2 &\simeq \frac{(p^2)^2}{\alpha}\left[1+\frac{e^2}{2\pi^2}p^2\beta(p^2,m^2)\right].\end{aligned}$$

As a result:

$$\begin{aligned}A'' &\simeq \frac{1}{p^2}\left\{1-\frac{e^2}{2\pi^2}\left[p^2\beta(p^2,m^2)+[(p.b)^2-p^2b^2]\chi(p^2,m^2)\right]\right\},\\ B'' &\simeq -\frac{e^2}{2\pi^2}\chi(p^2,m^2),\\ C'' &\simeq \frac{e^2}{2\pi^2}\frac{(p.b)}{(p^2)}\chi(p^2,m^2),\\ D'' &\simeq -\frac{1}{(p^2)^2}\left\{1-\alpha+\frac{e^2}{2\pi^2}\left[-p^2\beta(p^2,m^2)+p^2b^2\chi(p^2,m^2)\right]\right\},\\ E'' &\simeq \frac{e^2}{2\pi^2}\frac{1}{p^2}\gamma(p^2,m^2)).\end{aligned} \tag{36}$$

So, we have checked that as a consequence of the normalisation condition $\Pi^{(R)\mu\nu}|_{p^2=0} = 0$, the photon remains massless.

Further, if we expand the vacuum polarisation tensor around $p^2=0$, a new contribution to the Lamb-shift is obtained. In particular, considering a static configuration where $p^2=-\vec{p}^{\,2}$ we extract from (36) an approximate expression for the Coulomb interaction:

$$\frac{e^2}{\vec{p}^2}\left[1+\frac{e^2\vec{p}^2}{60\pi^2m^2}\left\{1-4\left[\frac{(\vec{p}.\vec{b})^2}{m^2\vec{p}^2}+\frac{b^2}{m^2}\right]\right\}\right].$$

Recalling that $|b/m| < 10^{-22}$ ([29]) we see that the effect of Lorentz breaking to the Lamb-shift is drastically tiny.

Of course, other contributions to the Lamb-shift effect coming from the vertex part should be considered but they will also be negligible (of the order $(\frac{b}{m})^2$).

## 4 Concluding Remarks

In this paper we have presented a complete 1-loop calculation of the vacuum polarisation tensor in the Lorentz and PCT violating QED defined by (1). This was done using the canonical perturbation theory and the P–V–R regularisation scheme. Let us remark that our calculation for the $\Pi_{bb}^{\mu\nu}$ allows us to correct a sign error in a previous calculation (formula (20) of [19]). We have checked gauge invariance and we comment on the appearance of an additional pole in the photon propagator. In order to avoid these difficulties and to follow the idea of spontaneous breaking of Lorentz symmetry in *a gauge invariant theory with a massless photon (QED) and a unique Lorentz breaking parameter*, among the possibilities of gauge invariant extra finite $\mathcal{O}(\hbar)$ terms, we analysed the particular one where we take as "normalisation conditions" the ones that stick to the tree level ones (absence of (2) coupling and an priori vanishing mass for the photon). The full photon propagator was then obtained for the first time through the Schwinger–Dyson equation allowing for a detailed analysis of its singular structure. No spurious pole appears and the photon indeed remains massless. The effect on the Lamb-shift coming from the dependence on $b$ in the vacuum polarisation appears in this case to be quite negligible.

Further investigations concerning the full 1-loop renormalisation of the extended QED as well as its thermal version are now in progress and will be reported in a forthcoming paper [30].

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